1) a) $\alpha'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$

$\Rightarrow |\alpha'(s)| = \sqrt{\frac{a^2}{c^2}\sin^2\frac{s}{c} + \frac{a^2}{c^2}\cos^2\frac{s}{c} + \frac{b^2}{c^2}} = \sqrt{\frac{a^2}{c^2} + \frac{b^2}{c^2}} = 1$

So s is arc-length //.

by $c^2 = a^2 + b^2$.

b) Since we are using arc length param., $T(s) = \alpha'(s)$ above and.

$k(s) = |\alpha''(s)|$.

$\alpha''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right)$.

$k(s) = |\alpha''(s)| = \sqrt{\frac{a^2}{c^4}\cos^2\frac{s}{c} + \frac{a^2}{c^4}\sin^2\frac{s}{c}} = \sqrt{\frac{a^2}{c^4}} = \frac{a}{c^2}$.

$N(s) = \frac{\alpha''(s)}{|\alpha''(s)|} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$

$N'(s) = \left(\frac{1}{c}\sin\frac{s}{c}, -\frac{1}{c}\cos\frac{s}{c}, 0\right)$

$\tau = -\langle N', B\rangle$

$= \langle N', T \times B\rangle = \det\begin{bmatrix} -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \\ \frac{1}{c}\sin\frac{s}{c} & -\frac{1}{c}\cos\frac{s}{c} & 0 \end{bmatrix} = \frac{b}{c^2}$.

So $k(s) = \frac{a}{c^2}$, $\tau(s) = \frac{b}{c^2}$ //.

c) $\cos\theta = \frac{\langle \alpha'(s), (0,0,1)\rangle}{|\alpha'(s)||1|} = \frac{\left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)\cdot(0,0,1)}{1\cdot 1}$

$= \frac{b}{c} = \text{const.}$

$\Rightarrow \theta$ is constant //.

2) By Q1, we know that if

$\beta = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c})$ is a circular helix,

$a, b > 0$, $a^2+b^2 = c^2$,

then β has curvature $k_\beta = \frac{a}{c^2}$, torsion $\tau_\beta = \frac{b}{c^2}$.

On the other hand, given k, τ constant, $k>0, \tau \neq 0$, let β be a circular helix parameterized as above with a, b s.t.

$$k = \frac{a}{a^2+b^2}, \quad \tau = \frac{b}{a^2+b^2}.$$

Solving for a, b explicitly, we see that setting

$$a = \frac{k^2}{k^2+\tau^2}, \quad b = \frac{\tau^2}{k^2+\tau^2}$$

means β will have curvature k, torsion τ.

Then by the fundamental theorem of local theory of curves, since α, β have the same curvature and torsion, α and β differ only by a rigid motion, which means α must be a circular helix.

Alternatively, we can integrate the Frenet-formulas to explicitly obtain a param. for α without using Fundamental Thm of curves.

By $N' = -\kappa T + \tau B$, we have

$$N'' = -\kappa' T - \kappa T' + \tau' B + \tau B'$$
$$= -\kappa' T - \kappa^2 N + \tau' B + \tau(-\tau N)$$
$$= -\underbrace{\cancel{\kappa'}}_{\text{constant}}{}^{\to 0} T + \cancel{\tau'}^{\to 0} B - \kappa^2 N - \tau^2 N$$
$$= -(\kappa^2 + \tau^2) N.$$

We recognize the solutions of this ODE as

$$N(s) = u \sin rs + v \cos rs \quad \text{for } r^2 = \kappa^2 + \tau^2,$$
some fixed $u, v \in \mathbb{R}^3$.

Note $N(0) = v$, and since $|N(s)|^2 = 1$, this implies $|v|^2 = 1$, i.e. v is a unit vector.

Also, differentiating, we have

$$N'(s)\Big|_{s=0} = (ru \cos rs - rv \sin rs)\Big|_{s=0} = ru$$

OTOH, we know

$$N'(0) = -\kappa T(0) + \tau B(0) \Rightarrow ru = -\kappa T(0) + \tau B(0).$$

So $r^2 |u|^2 = |-\kappa T(0) + \tau B(0)|^2$

$$= \kappa^2 \cancel{|T(0)|^2}^{1} - 2\kappa\tau \cancel{\langle T(0), B(0)\rangle}^{0} + \tau^2 \cancel{|B(0)|^2}^{1}$$
$$= \kappa^2 + \tau^2. \quad \Rightarrow |u|^2 = 1. \quad \text{So } u \text{ is also a unit vector.}$$

Also, $|N|^2=1$ implies

$$\cancelto{1}{|u|^2}\sin^2 rs + 2\sin rs\cos rs\langle u,v\rangle + \cancelto{1}{|v|^2}\cos^2 rs = 1$$

$$\Rightarrow 2\sin rs\cos rs\langle u,v\rangle \; 0 \quad \Rightarrow \langle u,v\rangle = 0.$$

Next, we can use $T' = \kappa N$, and write the ODE

$$\alpha''(s) = T'(s) = \kappa u\sin rs + \kappa v\cos rs.$$

So integrating once gives

$$\alpha'(s) = \frac{-\kappa}{r}u\cos rs + \frac{\kappa}{r}v\sin rs + w \quad \text{for some } w\in\mathbb{R}^3.$$

Integrating again gives

$$\alpha(s) = \frac{-\kappa}{r^2}u\sin rs - \frac{\kappa}{r^2}v\cos rs + ws + s_0 \quad \text{for some } s_0\in\mathbb{R}^3$$

Determine w: $\langle T,N\rangle = 0$ implies

$$0 = \langle \frac{-\kappa}{r}u\cos rs + \frac{\kappa}{r}v\sin rs + w,\; u\cos rs + v\sin rs\rangle$$

$$\Rightarrow \langle w, u\cos rs + v\sin rs\rangle = 0 \Rightarrow w \perp \operatorname{span}\{u,v\}.$$

Also, by $|T|^2=1$, we have that

$$1 = \frac{\kappa^2}{r^2}\cos^2 rs + \frac{\kappa^2}{r^2}\sin^2 rs + |w|^2 = \frac{\kappa^2}{r^2} + |w|^2$$

$$\Rightarrow |w|^2 = \frac{\tau^2}{r^2} \quad \Rightarrow |w| = \frac{|\tau|}{r}.$$

So finally we know that $\{-u, -v, w\frac{r}{|\tau|}\}$ is an orthonormal frame. Then after an orthonormal transformation taking this frame to the standard $\{e_1, e_2, e_3\}$, we get

$$\alpha(s) = \left(\frac{K}{r^2}\sin rs, \frac{K}{r^2}\cos rs, \frac{|\tau|}{r}s\right) + S_0$$

this is just translation by a point.

and we can clearly see that this is the parametrization of a circular helix.

3) WLOG, can take t to be arc-length param. of γ.
Since at $t=t_0$, $|\gamma(t)|$ is at local max, we have

$$\frac{d^2}{dt^2}\bigg|_{t=t_0} |\gamma(t)| \leq 0.$$

$$\text{LHS} = \frac{d^2}{dt^2}\bigg|_{t=t_0} \gamma(t)\cdot\gamma(t) = \frac{d}{dt}\bigg|_{t=0} 2\gamma'(t)\cdot\gamma(t)$$

$$= 2\gamma''(t)\cdot\gamma(t)\Big|_{t=t_0} + 2|\underbrace{\gamma'(t)}_{=1 \text{ by arc-length param.}}|^2\Big|_{t=t_0}$$

$$\Rightarrow 1 \leq |\gamma''(t_0)\cdot\gamma(t_0)|$$

$$\underset{\text{Cauchy Schwarz}}{\leq} |\gamma''(t_0)|\,\cancel{|\gamma(t_0)|}^{1} = |\gamma''(t_0)| = |k(t_0)|.$$

So we have $|k(t_0)| \geq 1$ as required. //

4) a) Rewriting in cartesian coordinates, we have

$\alpha(\theta) = (\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)$. $a \leq \theta \leq b$.

Then arc-length is given by $\int_a^b |\alpha'(\theta)|\, d\theta$.

$\alpha'(\theta) = (\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta, \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta)$.

$$
\begin{aligned}
|\alpha'(\theta)|^2 &= (\rho'(\theta))^2\cos^2\theta \cancel{- 2\rho'(\theta)\rho(\theta)\cos\theta\sin\theta} \\
&\quad + \rho^2(\theta)\sin^2\theta + (\rho'(\theta))^2\sin^2\theta \cancel{+ 2\rho'(\theta)\rho(\theta)\cos\theta\sin\theta} \\
&\quad + \rho^2(\theta)\cos^2\theta \\
&= (\rho'(\theta))^2 + \rho^2(\theta)
\end{aligned}
$$

$\Rightarrow$ arc-length $= \int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta$ //.

b) We'll use $k(\theta) = \dfrac{|\alpha'(\theta) \times \alpha''(\theta)|}{|\alpha'(\theta)|^3}$.

$$
\begin{aligned}
\alpha''(\theta) = (&\rho''(\theta)\cos\theta - 2\rho'(\theta)\sin\theta - \rho(\theta)\cos\theta, \\
&\rho''(\theta)\sin\theta + 2\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta).
\end{aligned}
$$

$$
\alpha'(\theta) \times \alpha''(\theta) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \rho'\cos\theta - \rho\sin\theta & \rho'\sin\theta + \rho\cos\theta & 0 \\ \rho''\cos\theta - 2\rho'\sin\theta - \rho\cos\theta & \rho''\sin\theta + 2\rho'\cos\theta - \rho\sin\theta & 0 \end{vmatrix}
$$

$$= \Big((\rho' \cos\theta - \rho \sin\theta)(\rho'' \sin\theta + 2\rho' \cos\theta - \rho \sin\theta)$$
$$- (\rho' \sin\theta + \rho\cos\theta)(\rho'' \cos\theta - 2\rho' \sin\theta - \rho\cos\theta) \Big) \hat{k}$$

$$= \Big(\cancel{\rho'\rho'' \cos\theta \sin\theta} + 2(\rho')^2 \cos^2\theta - \cancel{\rho\rho' \cos\theta \sin\theta} - \rho\rho'' \sin^2\theta$$
$$- \cancel{2\rho\rho' \sin\theta \cos\theta} + \rho^2 \sin^2\theta - \cancel{\rho'\rho'' \sin\theta \cos\theta} + 2(\rho')^2 \sin^2\theta$$
$$+ \cancel{\rho\rho' \sin\theta \cos\theta} - \rho\rho'' \cos^2\theta + \cancel{2\rho\rho' \cos\theta \sin\theta} + \rho^2 \cos^2\theta \Big) \hat{k}$$

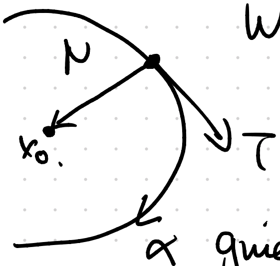
$$= \left(2(\rho')^2 - \rho\rho'' + \rho^2 \right) \hat{k}.$$

$$\Rightarrow |\alpha'(\theta) \times \alpha''(\theta)| = 2(\rho')^2 - \rho\rho'' + \rho^2$$

$$\Rightarrow k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\left((\rho')^2 + \rho^2\right)^{3/2}}$$

//.

5)



WLOG, param. α by arc-length s, and let the fixed point $x_0 = 0$.

Then the normal lines of α at s are given by $\beta_s(t) = \alpha(s) + tN(s)$.

We have that for all s, $\exists\, t(s)$ s.t.

$$\beta_s(t) = 0.$$

Then differentiating both sides in s

$$0 = \frac{d}{ds}\beta_s(t) = \frac{d}{ds}\left(\alpha(s) + tN(s)\right)$$

$$= T(s) + t'(s)N(s) + t(s)\left(-k(s)T(s) + \tau(s)B(s)\right)$$

$$= \left(1 - t(s)k(s)\right)T(s) + t'(s)N(s) + t(s)\tau(s)B(s).$$

Since $\{T(s), N(s), B(s)\}$ is a basis, equating components, we get

$t'(s) = 0 \Rightarrow t(s) \equiv \text{const}$. We can rule out $t(s) \equiv 0$ since then $\beta_s(t) \equiv \alpha(s)$ and condition will imply $\alpha(s)$ is a fixed point.

So WLOG, take $t > 0$ const.

$$\Rightarrow 0 = t\tau(s) \underset{\substack{\uparrow \\ t \neq 0.}}{\Rightarrow} \tau \equiv 0.$$ So α is contained in a plane.

and

$$1 - tk(s) = 0 \Rightarrow \underset{\substack{\uparrow \\ \text{radius of} \\ \text{curvature.}}}{\frac{1}{k}} = t = \text{const}.$$

Since radius of curvature is constant, and $\alpha(I)$ is in a plane, $\alpha(I)$ is contained in a circle. //

Alternatively, condition $\beta_s(t) = 0$ means at t,

$$0 = \langle \beta_s(t), \alpha'(s) \rangle = \langle \alpha(s), \alpha'(s) \rangle + t \underbrace{\langle N(s), \alpha'(s) \rangle}_{0}$$

since $N \perp T$.

$\Rightarrow \langle \alpha(s), \alpha'(s) \rangle \equiv 0 \Rightarrow \frac{d}{ds}|\alpha(s)| \equiv 0$.

$\Rightarrow |\alpha(s)| \equiv$ const. i.e. $\alpha(I)$ lies on a sphere.

Remains to show $\tau \equiv 0$ (in which case $\alpha(I)$ is on a plane $\cap$ sphere $=$ circle).

Condition also means $N(s) \parallel \alpha(s)$, so $N'(s) \parallel T(s)$,

so $$\frac{d}{ds}B(s) = \frac{d}{ds} T(s) \times N(s) = \underbrace{T'(s) \times N(s)}_{0} + \underbrace{T(s) \times N'(s)}_{0} = 0$$

$(T' \parallel N)$ $\qquad N' \parallel T \Rightarrow N' = \lambda T$ for some λ.

$\Rightarrow 0 = -\tau(s) N(s)$.

$\Rightarrow \tau(s) \equiv 0$.

Hence $\alpha(I)$ lies in a plane. So $\alpha(I)$ is contained in a circle //.